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On the Optimal Constant-Stress Acceleration Based on Competing Risks with Progressive Type-II Censoring for Lindley Distribution

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Abstract

In this study, We investigate competing risks and multiple levels constant-stress for accelerated life test using progressive type-II censoring. Assuming that the failure causes are independent and follow Lindley distribution. The optimization problem of the constant-stress for accelerated life test is investigated using two optimization criterions. Maximum likelihood estimate and the corresponding asymptotic confidence intervals are derived. Bayes estimate and credible confidence intervals are also obtained based on progressive type-II censoring. A real-world data examples are examined to illustrate the approaches employed in this study. Finally, simulation studies are executed to validate the estimates.

Keywords: constant-stress accelerated life test; maximum likelihood method; Bayesian method of estimation; credible confidence intervals; Markov chain Monte Carlo method; D-optimality criterion; A-optimality criterion.

1 Introduction

List of Abbreviations and Symbols

CSALT	:	Constant-stress accelerated life test.
PT-IIC	:	Progressive type-II censoring.
MLE	:	Maximum likelihood estimate.
LD	:	Lindley distribution.
BE	:	Bayes estimate.
CI	:	Confidence interval.
ALT	:	Accelerated life test.
SSALT	:	Step-stress accelerated life test.
CS	:	Censoring scheme.
MCMC	:	Markov chain Monte Carlo.
HRF	:	Hazard rate function.
SEL	:	Square error loss function.
LINXL	:	Linear exponential loss function.
MSE	:	Mean square error.
AE	:	Average estimate.
θ	:	The scale parameter of Lindley distribution.
λ	:	The acceleration factor from any transformed stress level.
$i = 1, \dots, k$	:	Number of constant stress levels.
$j = 1, \dots, m_i$	:	The numder of the observed test unit under S_i stress level.
$p = 1, \dots, r$	:	Number of failure causes.
I	:	The fisher information matrix of estimates.
$U(\Theta)$	:	The function of parameters θ_{01}, θ_{02} and λ.
c	:	The LINXL shape parameter.

Many products in the industrial field have gotten more consistent as a consequence of the rapid growth of equipments and their lives have become longer. It will be very hard to get the failure times of such products under normal operating conditions. As a result, new methodologies were created to accelerate the process of gathering sufficient failure data for high-reliability devices. The ALT aims to collect accurately modeled lifetime data that can be analyzed to learn more about products under normal conditions.

ALT is applied by submitting products to stringent conditions that accelerate failure occurrence, and can be conducted using many methods. Nelson [37] highlighted the benefits and drawbacks of these methods. The CSALT and the SSALT are the most frequently used methods. CSALT is widely used in different industries to evaluate the system and subsystem reliability, identify failure modes that need to be repaired, compare different manufacturers, etc. In CSALT, every unit is tested at a preset constant stress level, making it the simplest to apply. Assume n is the total test units available at k stress levels of CSALT. Also, assume that n_i represents the total number of units tested at the ith level of stress, where $\sum_{i=1}^{k} n_i = n$. ALT ends at a certain time point or when all test units fail. Many writers have written about CSALT, including Guan et al. [18] who studied the optimality of multiple CSALT for generalized exponential distribution and Jaheen et al. [23] who introduced the Bayesian inference in CSALT for the generalized exponential distribution under progressive censoring. Kim and Bai [25] proposed CSALT data analysis under two failure situations. Watkins and John [44] investigated the CSALT at one of the stress levels ended by type-II censoring. The inference related to CSALT based on geometric process under PT-IIC is discussed by Mohie El-Din et al. [34]. Abdel-Hamid [1] presented Burr type-XII distribution's Constant-partially-ALTs under PT-IIC. The CSALT plan, based on the chord of the nonlinear stress life relationship, was investigated by Gao et al. [13].

For SSALT, the applied stress on test units gradually increases at pre-determined periods or concurrently with the occurrence of a pre-determined failure numbers. Despite the fact that SSALT generates more failure data, the early changing time may alter the results of failures caused by greater stress levels. As a result, in SSALT, it is necessary to ascertain the ideal period for the stress levels to change. Many writers have written on SSALT, including Balakrishnan et al. [9] who obtained the point and interval estimation based on type-II censoring, Abu-Moussa et al. [36] who obtaind the expected Bayesian estimation for exponential model based on type-I hybrid censored data, Miller and Nelson [30] who proposed the optimum plans for the SSALT and Mohie El-Din et al. [35] who studied the parametric inference for the extension of exponential distribution under progressive type-II censoring. Bai et al. [7] investigated the ideal SSALTs under censoring. Gouno et al. [17] introduced a step-stress experiment under the optimal condition based on progressive censoring of type-I. Mohie El-Din et al. [12] applied the simple SSALT for analyzing the progressive first-failure of the Weibull data. Asgharzadeh et al. [4] estimated and predicted for proportional hazard family based on a simple SSALT model under type-II censored data.

Since all products are made up of multiple components that might fail for a variety of causes, these failure causes are investigated in order to draw correct inferences in survival analysis. In such circumstances, competing risks are explored in order to study the causes of failure for any product by a variable indicator that indicates the precise cause of failure. To analyze any set of data with competing risks, Cox [11] proposed the latent failure model. Several writers have explored competing risks with ALTs, including Mohie El-Din et al. [12] who stuided the step-stress partially accelerated life testing with competing risks based on the progressive type-II censoring, Han and Kundu [20] who obtained the statistical inference for the SSALT with competing risks for the generalized exponential distribution under type-I censoring, and Abu-Moussa et al. [22] who analyzed the progressive type-II competing risks data. Pascual [39] proposed planning ALT for independent Weibull and competing risks. Wu et al. [45] used progressive hybrid censoring by copula function to establish the inference for dependent competing risks with ALT.

In reliability tests, censoring is used when the experimental can not observe all the failure times for the sample. It also used to reduce the experiments time for collecting the lifetime data. The most often utilized censoring schemes for ALTs are type-I and type-II censoring. However, no test units under these two CSs are withdrawn during the experiment except at the last termination time point of the experiment.

As a result, progressive censoring is utilized to treat this objective and the focus on progressive censoring was rapidly increased in the last decades. PT-IIC is commonly employed in ALTs and analyzing enormously reliable data. For more specifics on PT-IIC, see the book of Balakrishnan and Aggarwala [10] which described the theory of progressive censoring with different applications. Several statistical conclusions were established under PT-IIC.

Abdel-Hamid [1] discussed the partially CSALT for a lifetimes distributed with Burr type-XII under PT-IIC. BE for extension of exponential distribution by MCMC method under PT-IIC data was introduced by Singh et al. [41]. Kumar et al. [26] established the product moments with its recurrence relations for Rayleigh distribution based on PT-IIC data. CSALT under PT-IIC is set by assuming S_0 is the stress level associated with typical usage, and k accelerated levels of stress with $S_1 < S_2 < \ldots < S_k$. Assume n_i units are tested at the constant stress level S_i, $i = 1, 2, \ldots, k$ with pre-fixed progressive CS $\{R_{ij}\}$, where $R_{ij} \geq 0$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, m_i$, $\sum_{j=1}^{m_i} R_{ij} + m_i = n_i$ and $m_i(\leq n_i)$, $i = 1, 2, \ldots, k$. Under the stress level S_i, $i = 1, 2, \ldots, k$, R_{i1} out of $n_i - 1$ surviving units are discarded from the test at the first occurrence of failure $t_{i1:m_i:n_i}$, while R_{i2} units from the remain $n_i - 2 - R_{i1}$ units are discarded at the second failure $t_{i2:m_i:n_i}$, and so on until the m_ith failure time $t_{im_i:m_i:n_i}$, where the test is terminated and all remain units $R_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} R_{ij}$ are dropped from the test.

LD is used to analyze data of failure times, principally, in the applications of modeling stress strength reliability. It was introduced by Lindley [27]. The merit of LD is the ability to model data of failure times with increasing hazard rates. It is one of the exponential family members and may be represented as a combination of gamma and exponential distributions, see Bakouch et al. [8]. It is also a good substitute for exponential failure time distributions, which don't show bathtub-shaped or unimodal failure rates. Mazucheli and Achcar [29] suggested the LD as a plausible substitute for the exponential and Weibull distributions.

Modeling the lifetime of any process or equipment is commonly done with LD, so it can be utilized in many different fields such as engineering, biology and medicine. Ghitany et al. [15] discussed LD and its application. Ghitany et al. [14] used LD for modeling death-rate studies. Shanker et al. [40] studied the two-parameter Lindley for modeling survival data. Valiollahi et al. [43] studied the prediction for LD under type-II right censored samples. Asgharzadeh et al. [5] introduced a statistical inference for LD model based on type-II censored data.

The LD's probability density function (PDF) of a random variable t is given by,

$$f(t) = \frac{\theta^2}{1+\theta}(1+t)e^{-\theta t}, \qquad t, \theta > 0, \tag{1}$$

while its cumulative distribution function (CDF) is given by,

$$F(t) = 1 - (1 + \frac{\theta t}{1+\theta})e^{-\theta t}, \qquad t, \theta > 0, \tag{2}$$

and the hazard rate function (HRF),

$$h(t) = \frac{\theta(1+t)}{1+(1+t)\theta}, \qquad t, \theta > 0. \tag{3}$$

Please refer in Figure 1.

The innovation proposed in this paper is to study the CSALT under progressive type-II competing risks data follow the Lindely lifetime distribution. Our work is motivated by the leaking of the studies which combine the ALT with competing risks under censoring framework. The paper is organized as follows, the description of the CSALT from LD with competing risks model with test procedures are displayed in Section 2. MLEs by Newton-Raphson iteration method and asymptotic CIs are described in Section 3. Section 4 describes the BEs by MCMC method and credible CIs based on the informative and non-informative priors In Section 5, Two optimization criteria are proposed to investigate the optimal transformed stress levels. Two sets of real data are provided as examples in Section 6. The simulation results are shown in Section 7, based on the suggested estimation techniques. Finally, some conclusions are discussed in Section 8.

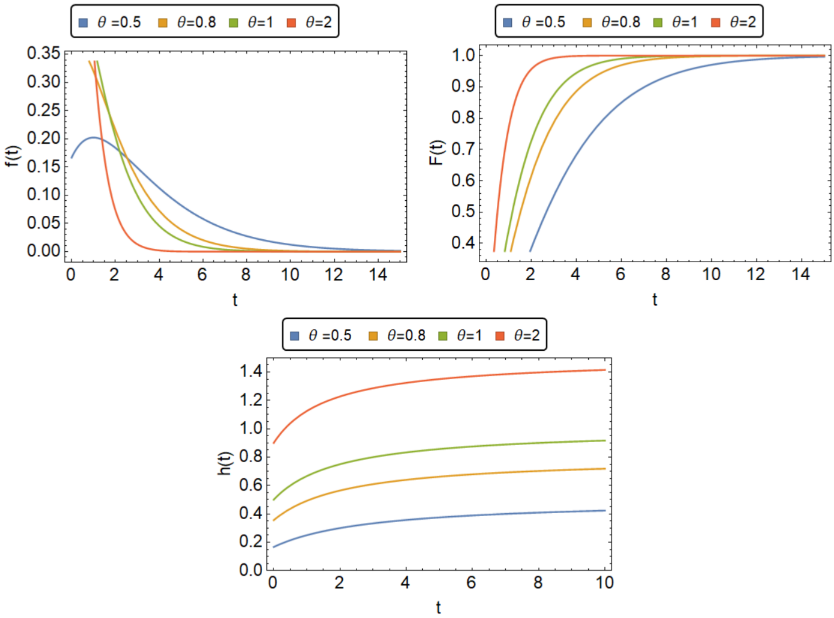


Figure 1: The PDF, CDF and HRF of LD.

2 Model Description and Test Assumptions

The used procedures through out this paper are stated as following:

1. For the CSALT, under S_i, $i = 1, \ldots, k$ stress level, the T_i failure time is fitted by LD.
2. The function which links between the lifetime parameter θ and the applied stress S can be expressed as any of these rules:
   - Arrhenius relation: $\ln(\theta) = a + \dfrac{b}{-S}$, $b > 0$, which used when S is a temperature variable.
   - Inverse power relation: $\ln(\theta) = a + b[\ln(S)]$, $b > 0$, which used when S is a voltage variable.
   - Exponential relation: $\ln(\theta) = a + bS$, $b > 0$, which used when S is a weathering variable.

   Refer to Nelson [37] for the statistical models, test plans and data analyses for estimating product reliability from accelerated tests. Thus, we suppose that S_i and θ_i have the following connected function,

$$\ln(\theta_i) = a + b\Omega_i, \tag{4}$$

where $i = 0, 1, \ldots, k$, a and b are positive unknown physical parameters, and an increasing function of S, $\Omega_i = \Omega(S_i) = \dfrac{-1}{S_i}$. From (4), the parameter θ_i is written as,

$$\theta_i = \theta_0 e^{b(\Omega_i - \Omega_0)} = \theta_0 \lambda^{h_i}, \qquad i = 0, 1, \ldots, k, \tag{5}$$

where θ_0 is the normal usage level S_0 parameter of the LD, $\lambda = \exp\left\{b(\Omega_1 - \Omega_0)\right\} = \dfrac{\theta_1}{\theta_0} > 1$ represents the factor of acceleration from S_0 to S_1 and any changed stress level

$$h_i = \frac{\Omega_i - \Omega_0}{\Omega_1 - \Omega_0}, \tag{6}$$

where $1 \leq h_i < \infty$, $i = 1, 2, \ldots, k$.

3. Assume that we are able to observe the failed test unit's failure time and cause at the i_{th} stress level. Thus, we consider that the observed lifetime of this failed test unit is the smallest one of the latent failure times. Let, the latent failure times $x_{ij1}, x_{ij2}, \ldots, x_{ijr}$ are independent. Then, the failure time of the j_{th} test unit is $T_{ij} = \min\left\{x_{ij1}, x_{ij2}, \ldots, x_{ijr}\right\}$. Hence, the joint PDF and CDF of failure cause p and failure time of the j_{th} unit at the i_{th} stress level are given, respectively,

$$f(t_{ij}, p) = \frac{\theta_{0p}^2}{1+\theta_{0p}}\left(1+t_{ij}\right)e^{-\theta_{0p}t_{ij}}, \qquad t_{ij}, \theta_{0p} > 0, \tag{7}$$

$$F(t_{ij}, p) = 1 - \left(1 + \frac{\theta_{0p}t_{ij}}{1+\theta_{0p}}\right)e^{-\theta_{0p}t_{ij}}, \qquad t_{ij}, \theta_{0p} > 0. \tag{8}$$

4. Assuming that failure causes are independent, the following indicator can be used to indicate the latent failure time can be expressed, for $j = 1, 2, \ldots, m_i$, as follows,

$$c_{ijp} = \begin{cases} 1, & \text{if} \quad T_{ij} = x_{ijp}, \\ 0, & \quad \text{otherwise.} \end{cases} \tag{9}$$

5. Let $t_{ij} = t_{ij:m_i:n_i}$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m_i$ are the resulted failure times under S_i stress level for the failure cause p, where $p = 1, 2, \ldots, r$. Hence, the likelihood function of θ_{0p} and λ under PT-IIC is presented by,

$$L(\theta_{0p}, \lambda) = \prod_{p=1}^{r}\prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} f_{T_{ip}}^{c_{ijp}}(t_{ij})\left[1 - F_{T_{ip}}(t_{ij})\right]^{R_{ij}}, \tag{10}$$

where

$$C_i = n_i\left(n_i - 1 - R_{i1}\right)\left(n_i - 2 - R_{i1} - R_{i2}\right)\cdots\left(n_i - m_i + 1 - \sum_{j=1}^{m_i-1} R_{ij}\right).$$

Since $S_{T_{ip}}(t_{ij}) = 1 - F_{T_{ip}}(t_{ij})$ and $f_{T_{ip}}(t_{ij}) = h_{T_{ip}}(t_{ij})S_{T_{ip}}(t_{ij})$, likelihood function is expressed as following,

$$L(\theta_{0p}, \lambda) = \prod_{p=1}^{r}\prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} h_{T_{ip}}^{c_{ijp}}(t_{ij})\left[S_{i_p}(t_{ij})\right]^{(R_{ij}+1)}. \tag{11}$$

3 Maximum Likelihood Estimation

MLEs of the parameters are obtained by utilizing the above procedures. Assuming the failure causes number is $(r = 2)$ and by substituting in (7), (8) and (11). The likelihood function of θ_{01},

θ_{02} and λ is given as following,

$$L(\theta_{01},\theta_{02},\lambda|t_{ij}) \propto \prod_{i=1}^{k}\prod_{j=1}^{m_i}\Bigg[\left(\frac{(1+t_{ij})\theta_{01}\lambda^{h_i}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right)^{c_{ij1}}\left(\frac{(1+t_{ij})\theta_{02}\lambda^{h_i}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)^{c_{ij2}} \left(\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)e^{-\theta_{01}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1} \left(\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)e^{-\theta_{02}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\Bigg]. \tag{12}$$

Thus, log likelihood function is presented as,

$$\begin{aligned}\ell &= \log L(\theta_{01},\theta_{02},\lambda|t_{ij})\\ &\propto \sum_{i=1}^{k}\sum_{j=1}^{m_i}\Bigg[c_{ij1}\big(\log\left((1+t_{ij})\theta_{01}\lambda^{h_i}\right)-\log\left(1+(1+t_{ij})\theta_{01}\lambda^{h_i}\right)\big)\\ &\quad + c_{ij2}\Big(\log\left((1+t_{ij})\theta_{02}\lambda^{h_i}\right)-\log\left(1+(1+t_{ij})\theta_{02}\lambda^{h_i}\right)\Big)\\ &\quad + (R_{ij}+1)\left(\log\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)+\log\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)-(\theta_{01}+\theta_{02})\lambda^{h_i}t_{ij}\right)\Bigg].\end{aligned} \tag{13}$$

By differentiating the log-likelihood function in terms of each parameter before being equated to zero. Here, we obtain three nonlinear equations with three unknown parameters θ_{01},θ_{02} and λ as follows,

$$\begin{aligned}\frac{\partial\ell}{\partial\theta_{01}} = J_1(\theta_{01}) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\Bigg[c_{ij1}\left(\frac{1}{\theta_{01}}-\frac{(1+t_{ij})\lambda^{h_i}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right)\\ +(R_{ij}+1)\left(\frac{\lambda^{h_i}t_{ij}(1+\theta_{01}\lambda^{h_i}-\lambda^{h_i})}{\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)\left(1+\theta_{01}\lambda^{h_i}\right)^2}-\lambda^{h_i}t_{ij}\right)\Bigg]=0,\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\theta_{02}} = J_2(\theta_{02}) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\Bigg[c_{ij2}\left(\frac{1}{\theta_{02}}-\frac{(1+t_{ij})\lambda^{h_i}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)\\ +(R_{ij}+1)\left(\frac{\lambda^{h_i}t_{ij}(1+\theta_{02}\lambda^{h_i}-\lambda^{h_i})}{\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)(1+\theta_{02}\lambda^{h_i})^2}-\lambda^{h_i}t_{ij}\right)\Bigg]=0,\end{aligned} \tag{15}$$

$$\frac{\partial \ell}{\partial \lambda} = J_3(\lambda) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\Bigg[c_{ij1}\left(\frac{h_i}{\lambda} - \frac{h_i(1+t_{ij})\theta_{01}\lambda^{h_i-1}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right) + c_{ij2}\left(\frac{h_i}{\lambda} - \frac{h_i(1+t_{ij})\theta_{02}\lambda^{h_i-1}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)$$
$$+ (R_{ij}+1)\left(\frac{h_i\lambda^{h_i-1}\theta_{01}t_{ij}}{\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)(1+\theta_{01}\lambda^{h_i})^2}\right.$$
$$\left.+\frac{h_i\lambda^{h_i-1}\theta_{02}t_{ij}}{\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)(1+\theta_{02}\lambda^{h_i})^2} - (\theta_{01}+\theta_{02})h_i\lambda^{h_i-1}t_{ij}\right)\Bigg] = 0. \tag{16}$$

These three nonlinear equations are very difficult to be solved in closed form. Therefore, an iterative method like Newton-Raphson is utilized to get a numerical solution for such nonlinear system.

3.1 Existence and uniqueness of MLEs

In this subsection, we discuss the existence and uniqueness of the MLEs that are obtained by solving (14), (15), and (16). Since these equations are nonlinear and the MLEs can't obtained in an exact form, then the existence and uniqueness of the MLEs of the parameters θ_{01}, θ_{02} and λ can't be proved analytically. So, it will be proved numerically and graphically by assuming that there are two constant parameters and the third one is variable. for instance, if we assumed that θ_{02} and λ are constants, then $J_1(\theta_{01})$ in (14) is a function in one variable, it is easy to prove that the function has a unique solution by showing graphically that the derivative of $J_1(\theta_{01})$ is a negative function which means that $J_1(\theta_{01})$ is a decreasing function intersecting the horizontal axis one time which is the unique solution of θ_{01} satisfies $J_1(\theta_{01}) = 0$ and negative value as θ_{01} tends to ∞. The same situation can be applied for the parameters θ_{02} under the function $J_2(\theta_{02})$ and λ under the function $J_3(\lambda)$. Many authors have used this method of proving the existence and uniqueness when having a nonlinear equations of MLEs. Abu-Moussa et al. [3] applied this method to prove the uniqueness of MLEs of the two model parameters since the two nonlinear equations couldnot be proved analytically and were proved numerically by the same applied method.

Illustrative Example

An illustrative example using a simulated data is applied to apply the graphical and numerical prove of the existence and uniqueness of the MLEs. Assuming $n_1 = 29$, $m_1 = 25$, $n_2 = 16$, $m_2 = 13$, $n_3 = 13$, $m_3 = 11$, and $k = 3$, we generate a sample of 3 stress levels using real values of parameters $\theta_{01} = 0.4$, $\theta_{02} = 0.5$, and $\lambda = 1.2$. Table 1 shows the simulated data required for this example.

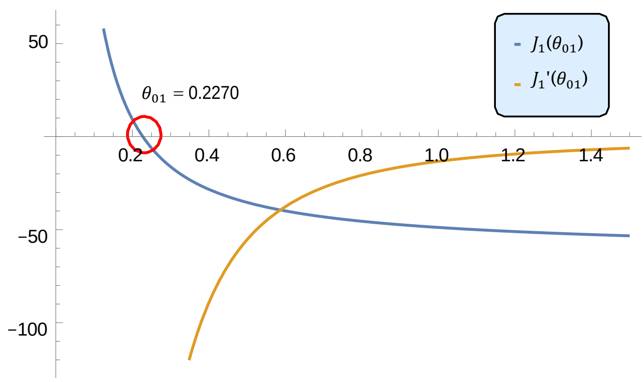
Table 1: Simulated data sets for 3 levels of CSALT.

Level 1	0.01124	0.14838	0.21342	0.22578	0.36963	0.39090	0.43537	0.46058	0.60343
	0.63179	0.64344	0.65612	0.73252	1.01652	1.10730	1.15468	1.26533	1.5258
	1.75590	1.91140	2.01859	2.62523	4.39639	5.74587	5.81403		
Level 2	0.29965	0.51839	0.52253	0.60831	0.67654	1.11693	1.19244	1.31814	1.37079
	1.57742	1.65436	1.70910	3.80188					
Level 3	0.21477	0.22654	0.33771	0.53482	0.693842	0.93051	1.13398	1.34516	2.21883
	3.25939	4.03266							

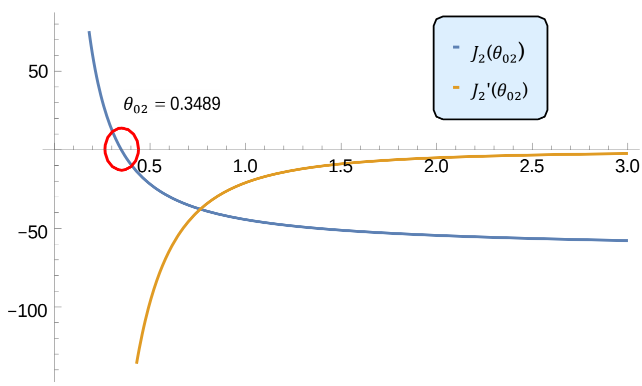
where

$$R_1 = \{0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0\},$$
$$R_2 = \{0,0,0,0,1,1,1,0,0,0,0,0,0\}, \quad \text{and}$$
$$R_3 = \{0,0,0,0,1,1,0,0,0,0,0\}.$$

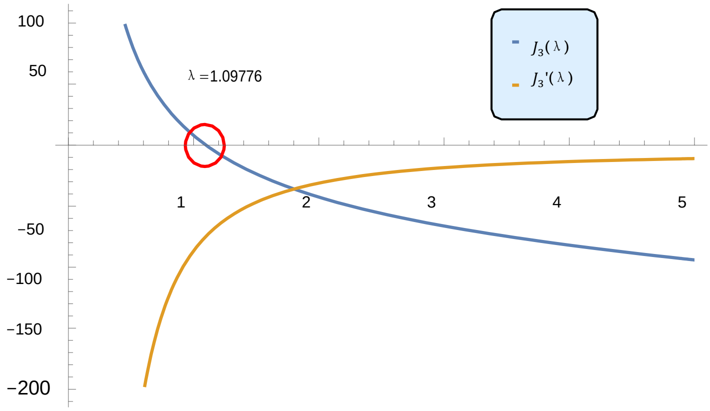
Figures 2(a), 2(b) and 2(c) show the behaviour of $J_1(\theta_{01})$, $J_2(\theta_{02})$ and $J_3(\lambda)$ with their derivatives. Its obvious that we have only one ML estimate value for each parameter when the other two parameters are considered to be constant.



(a) $J_1(\theta_{01})$, $J_1'(\theta_{01})$.



(b) $J_2(\theta_{02})$, $J_2'(\theta_{02})$.



(c) $J_3(\lambda)$, $J_3'(\lambda)$.

Figure 2: The existence and uniqueness of the likelihood estimates of each variable when the two other variables are given or remaining constant.

3.2 Asymptotic confidence intervals

Normal asymptotic CIs of the three parameters can be obtained using asymptotic distribution which is introduced by Miller [32]. Given a random quantity υ, the asymptotic distribution of the MLEs of υ represents θ_{01}, θ_{02} or λ, is given as follows,

$$\left((\hat{\theta_{01}} - \theta_{01}), (\hat{\theta_{02}} - \theta_{02}), (\hat{\lambda} - \lambda)\right) \sim \mathbf{N}\left(0, I^{-1}\right),$$

where I is the Fisher information matrix of estimates and expressed as,

$$I\left(\widehat{\theta_{01}}, \widehat{\theta_{02}}, \widehat{\lambda}\right) = \begin{pmatrix} -\dfrac{\partial^2 \ell}{\partial \theta_{01}^2} & -\dfrac{\partial^2 \ell}{\partial \theta_{01} \partial \theta_{02}} & -\dfrac{\partial^2 \ell}{\partial \theta_{01} \partial \lambda} \\ -\dfrac{\partial^2 \ell}{\partial \theta_{01} \partial \theta_{02}} & -\dfrac{\partial^2 \ell}{\partial \theta_{02}^2} & -\dfrac{\partial^2 \ell}{\partial \theta_{02} \partial \lambda} \\ -\dfrac{\partial^2 \ell}{\partial \theta_{01} \partial \lambda} & -\dfrac{\partial^2 \ell}{\partial \theta_{02} \partial \lambda} & -\dfrac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}. \tag{17}$$

The asymptotic $100\ (1 - \varphi)\%$ two sided CI of υ is represented by,

$$(\hat{\upsilon}_l,\ \hat{\upsilon}_u) = \hat{\upsilon} \pm Z_{\varphi/2}\sqrt{\sigma_{ii}}, \qquad i = 1, 2, 3, \tag{18}$$

where σ_{ii} is the variance of υ derived from the diagonal of I^{-1}, and Z_{φ} is $100\varphi - th$ percentile of regular normal distribution.

4 Bayes Estimation

BEs of the three parameters are derived by using both of SEL and LINXL functions. LINXL means the linear exponential loss function which is applied in analyzing the statistical estimation and prediction problem which rises exponentially on one side of zero and almost linearly on the other side of zero. It is used in both overestimation and underestimation problems. LINXL function is applied in analyzing statistical estimation since it rises exponentially. It can be applied in both overestimation and underestimation problems. The SEL function is given by,

$$L(\hat{\theta}, \theta) \propto (\hat{\theta} - \theta)^2. \tag{19}$$

The Bayes estimate relative to the SE loss function $\hat{\theta}_{BS}$ is given by,

$$\hat{\theta}_{BS} = E(\theta) = \int_{\theta} \theta \, \pi^*(\theta|\mathbf{t}) \, d\theta. \tag{20}$$

The LINXL function $L_{LIN}(\hat{\theta}, \theta)$ is introduced as,

$$L_{LIN}(\hat{\theta}, \theta) \propto e^{c(\hat{\theta}-\theta)} - c(\hat{\theta} - \theta) - 1, \tag{21}$$

where $c \neq 0$ is the shape parameter of LINXL function, $\hat{\theta}$ is an estimate of θ. The sign and magnitude of the shape parameter (c) express respectively the direction and degree of symmetry. When $c > 0$, the overestimate is more serious than underestimate, and vice-versa. For c close to zero, the LINXL is approximately SE loss and then almost symmetric. The Bayes estimator of θ under the LINXL function is defined by,

$$\hat{\theta}_{BL} = \frac{-1}{c} \log\left(E(e^{-c\theta})\right) = \frac{-1}{c} \log\left(\int_{\theta} e^{-c\theta} \, \pi^*(\theta|\mathbf{t}) \, d\theta\right). \tag{22}$$

For more details about LINXL function, see Khatun [24] since they have explained how the LINXL works in terms of changing the shape parameter and the error function through practical or detail explanations. BEs are obtained by using informative priors. The prior distributions are assumed to be gamma owing to their flexibility, and because they accommodate a variety of shapes reflected in prior beliefs. Miller [31] presented a Bayesian analysis of shape, scale, and mean of the two-parameters gamma distribution and simplifications of the numerical analysis of posterior distributions. Also, for more information about gamma prior see Moala [33] since he derived distinct prior distributions in a Bayesian inference of the two-parameters gamma distribution. Many authors used the gamma prior to obtain BEs, Goltong and Doguwa [16] discussed the BEs of the Behrens-Fisher problem by using a Gamma Prior. Hasan and Baizid [21] studied the BEs under different loss functions by using gamma prior for exponential distribution.

Assuming that θ_{01}, θ_{02} and λ are independent parameters with gamma priors, then,

$$\pi_1(\theta_{01}) \propto \theta_{01}^{a_1-1} e^{-\theta_{01} a_2}, \qquad \theta_{01} > 0, \quad a_1, a_2 > 0, \tag{23}$$

$$\pi_2(\theta_{02}) \propto \theta_{02}^{b_1-1} e^{-\theta_{02} b_2}, \qquad \theta_{02} > 0, \quad b_1, b_2 > 0, \tag{24}$$

$$\pi_3(\lambda) \propto \lambda^{g_1-1} e^{-\lambda g_2}, \qquad \lambda > 0, \quad g_1, g_2 > 0. \tag{25}$$

The values of hyper parameters a_i, b_i and g_i, $i = 1, 2$, are predetermined by the experimenter. The values of the prior parameters (a_1,a_2), $(b1,b_2)$ and (g_1,g_2) are obtained by using the mean and variance of gamma distribution. The non-informative priors are a special case from the informative priors in (23)–(25) when $a_i = b_i = g_i = 0$, $i = 1, 2$.

From (23)–(25), the joint prior of the three parameters is given by,

$$\pi(\theta_{01}, \theta_{02}, \lambda) \propto \theta_{01}^{a_1-1} \theta_{02}^{b_1-1} \lambda^{g_1-1} e^{-\theta_{01} a_2 - \theta_{02} b_2 - \lambda g_2}, \qquad \theta_{01}, \theta_{02} > 0, \qquad \lambda > 1. \tag{26}$$

From (12) and (26), the joint posterior distribution is given by,

$$\begin{aligned}\pi^*(\theta_{01},\theta_{02},\lambda|t) &\propto L(\theta_{01},\theta_{02},\lambda)\pi(\theta_{01},\theta_{02},\lambda)\\ &\propto \theta_{01}^{a_1-1}\theta_{02}^{b_1-1}\lambda^{g_1-1}e^{-\theta_{01}a_2-\theta_{02}b_2-\lambda g_2}\\ &\prod_{i=1}^{k}\prod_{j=1}^{m_i}\Bigg[\left(\frac{(1+t_{ij})\theta_{01}\lambda^{h_i}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right)^{c_{ij1}}\left(\frac{(1+t_{ij})\theta_{02}\lambda^{h_i}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)^{c_{ij2}}\\ &\left((1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}})e^{-\theta_{01}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\\ &\left((1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}})e^{-\theta_{02}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\Bigg].\end{aligned} \tag{27}$$

BEs for the function of $U(\Theta) = U(\theta_{01}, \theta_{02}, \lambda)$ using SEL and LINXL functions are obtained by,

$$\widetilde{U}_{SE}(\Theta) = E(U(\Theta)) = \int_{\Theta} U(\Theta)\pi^*(\Theta|\mathbf{t})d\Theta, \tag{28}$$

and

$$\begin{aligned}\widetilde{U}_{LINX}(\Theta) &= -\frac{1}{c}\ln\left[E(e^{-c\,U(\Theta)})\right]\\ &= -\frac{1}{c}\ln\left[\int_{\Theta} e^{-c\,U(\Theta)}\pi^*(\Theta|\mathbf{t})d\Theta\right],\end{aligned} \tag{29}$$

where the LINXL function's shape argument is $c \neq 0$ and the expected value is $E(.)$. These expectations in (28) and (29) cannot be computed explicitly. Therefore, to approximate these expectations, the MCMC approach is applied.

4.1 MCMC approximation

MCMC approximation is applied through this subsection to approximate the BEs of θ_{01}, θ_{02} and λ by generating a samples from the posterior distribution. Metropolis-Hastings algorithm is applied as shown in Algorithm (1). The starting values are represented by $(\theta_{01}, \theta_{02}, \lambda)$ say $\theta_{01}^{(0)}$, $\theta_{02}^{(0)}$ and $\lambda^{(0)}$ in most cases it chosen as the maximum likelihood estimates.

There may be some worry that an MCMC algorithm's starting values could distort outcomes even if it eventually approaches this equilibrium distribution because it is rarely initialized from its invariant distribution. In order to make up for this, a burn-in phase is frequently used, during which the first M samples are deleted. M is typically set to be large enough such that the chain has achieved its stationary regime by this point. Convergence of MCMC approximation assures that we have sufficiently many samples to approximate the posterior distribution. For more details about Metropolis-Hasting algorithm; see Updhyay and Gupta [42] where Bayes estimators were obtained by using MCMC algorithm under the balanced SE loss function and BEs were compared with their corresponding MLEs based on Monte Carlo simulation. From (27), the conditional postrior distributions of θ_{01}, θ_{02} and λ are given by:

$$P_1(\theta_{01}|\theta_{02},\lambda) \propto \theta_{01}^{a_1-1}e^{-\theta_{01}a_2}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\Bigg[\left(\frac{(1+t_{ij})\theta_{01}\lambda^{h_i}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right)^{c_{ij1}} \left(\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)e^{-\theta_{01}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\Bigg], \tag{30}$$

$$P_2(\theta_{02}|\theta_{01},\lambda) \propto \theta_{02}^{b_1-1}e^{-\theta_{02}b_2}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\Bigg[\left(\frac{(1+t_{ij})\theta_{02}\lambda^{h_i}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)^{c_{ij2}} \left(\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)e^{-\theta_{02}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\Bigg], \tag{31}$$

$$P_3(\lambda|\theta_{01},\theta_{02}) \propto \lambda^{g_1-1}e^{-\lambda g_2}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\Bigg[\left(\frac{(1+t_{ij})\theta_{01}\lambda^{h_i}}{1+(1+t_{ij})\theta_{01}\lambda^{h_i}}\right)^{c_{ij1}}\left(\frac{(1+t_{ij})\theta_{02}\lambda^{h_i}}{1+(1+t_{ij})\theta_{02}\lambda^{h_i}}\right)^{c_{ij2}} \left(\left(1+\frac{\theta_{01}\lambda^{h_i}t_{ij}}{1+\theta_{01}\lambda^{h_i}}\right)e^{-\theta_{01}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1} \left(\left(1+\frac{\theta_{02}\lambda^{h_i}t_{ij}}{1+\theta_{02}\lambda^{h_i}}\right)e^{-\theta_{02}\lambda^{h_i}t_{ij}}\right)^{R_{ij}+1}\Bigg]. \tag{32}$$

Given the complexity of the conditional posteriors of θ_{01}, θ_{02} and λ, it is not feasible to constringe them analytically to well-known distributions. Therefore, to generate from these distributions some random samples, we turn to Metropolis-Hastings algorithm. This algorithm is used to obtain the BEs of $U = U(\theta_{01}, \theta_{02}, \lambda)$ under SE and LINXL functions:

Algorithm (1)

1. Start by the initial points of $(\theta_{01}, \theta_{02}, \lambda)$ say $(\theta_{01}^{(0)}, \theta_{02}^{(0)}, \lambda^{(0)})$.
2. Start with $i = 1$.
3. Use Metropolis-Hasting algorithm to generate $\theta_{01}^{(i)}$, $\theta_{02}^{(i)}$ and $\lambda^{(i)}$ from (30), (31) and (32) respectively.
4. Let $i = 1 + i$.
5. Perform Steps $(2-4)$ for N times.
6. The approximated mean of e^{-cU} and U can be obtained as,

$$E(e^{-cU}) = \frac{1}{N-M}\sum_{i=M+1}^{N}\exp\{-cU(\theta_{01}^{(i)}, \theta_{02}^{(i)}, \lambda^{(i)})\}, \tag{33}$$

$$E(U) = \frac{1}{N-M}\sum_{i=M+1}^{N}U(\theta_{01}^{(i)}, \theta_{02}^{(i)}, \lambda^{(i)}), \tag{34}$$

   where the burn-in period is M.
7. End.

4.2 Credible confidence intervals

In this subsection, an intervel estimation for the model parameters are established using the Credible CIs. The $100(1-\varphi)\%$ Bayesian credible or posterior interval for a random variable υ is defined as follows:

Considering that υ lies in the interval where,

$$\begin{aligned} p(l \leq \upsilon \leq u) &= \int_l^u \pi^*(\upsilon|\mathbf{t})d\upsilon \\ &= 1-\varphi. \end{aligned}$$

Now, the credible CIs of θ_{01}, θ_{02} and λ are established through the subsequent algorithm:

Algorithm (2)

1. Perform Steps $(1-6)$ from Algorithm (1).
2. Arrange all generated values in ascending order as $\left\{\tilde{\theta}_{01}^{[1]}, \tilde{\theta}_{01}^{[2]}, \ldots, \tilde{\theta}_{01}^{[N]}\right\}$, $\left\{\tilde{\theta}_{02}^{[1]}, \tilde{\theta}_{02}^{[2]}, \ldots, \tilde{\theta}_{02}^{[N]}\right\}$ and $\left\{\tilde{\lambda}^{[1]}, \tilde{\lambda}^{[2]}, \ldots, \tilde{\lambda}^{[N]}\right\}$.
3. End.

Then, $100\ (1-\varphi)\%$ credible CI of υ is given as,

$$(\tilde{\upsilon}_l,\ \tilde{\upsilon}_u) = \left(\tilde{\upsilon}^{[\varphi N/2]},\ \tilde{\upsilon}^{[(1-\varphi/2)N]}\right), \text{since } \upsilon \text{ represents } \theta_{01},\ \theta_{02} \text{ or } \lambda. \tag{35}$$

5 Optimal Stress Levels

It is very important to designing an ALT to estimate the reliability characteristics of any product at the normal use conditions. Over the last years, the issue of choosing optimal designs of any ALT has got a lot of attention in the reliability researches. A lot of researches have focused on developing the optimal designs of ALTs, see, Bai and Chung [6] for optimal designs for PALTs in which items were run at both accelerated and use conditions until a predetermined time are considered. Gouno et al. [17] considered a k-SSALT with equal duration steps and censoring was allowed at each change stress point. Bai et al. [7] obtained the optimum test plans to minimize the asymptotic variance of the MLE of the mean life at a design stress and the optimum failure-step stress test plans were obtained.

Yang [47] introduced an optimal design for a four level CSALT. Abdel-Hamed and AL-Hussaini [2] discussed the issue of optimality based on the progressively type-I censored SSALT data for the generalized Pareto distribution. The time and cost constrain es are considered in the study of optimality of CSALTs and SSALTs that executed by Han [19] based on the exponential distribution. Concerning the optimal designs of ALTs in competing risks model, Liu and Qiu [28] presented some numerical results of SSALT planning using Compatible with independent Weibull competing risks. Pascual [38] studied the optimal plans of the CSALT with independent competing risks based on Weibull distribution. The planning of CSALT was introduced by Wu and Huang [46] when the competing risks of exponential lifetime distribution are exist. Most studies in these papers provided numerical results only due to the complexity of the work.

In this section, D-optimality and A-optimality criterion are proposed by using the Fisher information matrix. Determining the optimal transformed stress levels h_i, $i = 1, 2, \ldots, k$ is proposed for CSALT using PT-IIC data from LD. Assuming that $k = 2$ stress levels, (h_1, h_2) in the CSALT, the optimal stress level of h_2 should be calculated since the smallest transformed stress level is fixed at $h_1 = 1$.

5.1 D-optimality criterion

The D-optimality criterion is a test plan which maximizes the determinant of Fisher information matrix. Maximizing the determinant is equivalent to minimizing the joint confidence region of the parameters. Hence, from the Fisher information matrix I for the MLEs in (17), the determinant of this matrix can be obtained as,

$$I = -\frac{\partial^2 \ell}{\partial \theta_{01}^2}\frac{\partial^2 \ell}{\partial \theta_{02}^2}\frac{\partial^2 \ell}{\partial \lambda^2} + \frac{\partial^2 \ell}{\partial \theta_{01}^2}\left(\frac{\partial^2 \ell}{\partial \theta_{02}\partial \lambda}\right)^2 + \left(\frac{\partial^2 \ell}{\partial \theta_{01}\partial \lambda}\right)^2 \frac{\partial^2 \ell}{\partial \theta_{02}^2}. \tag{36}$$

Now, the optimal transformed value of the second stress level h_2^* is calculated by,

$$\text{Maximize}\left\{\det\left(\mathbf{I}(\hat{\theta_{01}}, \hat{\theta_{02}}, \hat{\lambda})\right)\right\}. \tag{37}$$

5.2 A-optimality criterion

Another important criterion is the A-optimality criterion. It depends on minimizing trace of the asymptotic varaince-covaraince matrix. The A-optimality criterion aims to minimize sum of the main diagonal elements of Fisher information matrix's inverse. Then, the optimal transformed stress level h_2^* is calculated by:

$$\text{Minimize}\left\{tr\left(I^{-1}(\hat{\theta_{01}}, \hat{\theta_{02}}, \hat{\lambda})\right)\right\}. \tag{38}$$

6 Illustrative Examples

6.1 Example 1

This example is introduced to show the proposed methods. Nelson [37], page 393, has introduced a data set for failure times of class-H insulation system in motors. As the design temperature was 180^oC, the insulation systems were tested at high temperatures of 190^oC, 220^oC, 240^oC and 260^oC. Assuming that the causes of failure are independent as they occur on separate parts of the insulation system, there were three causes of failure, the Turn, Phase and Ground. However, only Turn (Cause 1) and Ground (Cause 2) failure causes are considered here, and we use the data from 220^oC and 240^oC. These data are represented in Table 2.

Table 2: Class-H insulation system failure time data with its cause of failure.

220^oC		240^oC	
X_{i1}	Cause	X_{i1}	Cause
1764	1	1175	2
2436	1	1175	2
2436	2	1521	1
2436	1	1569	1
2436	2	1617	1
2436	1	1665	1
3108	1	1665	1
3108	1	1713	1
3108	1	1761	1
3108	1	1953	1

Then, Arrhenius model was used to illustrate relationship describing the stress levels and the mean lifetime when temperature represented the accelerating factor. The Arrhenius model is considered the most common applied life-stress relationship for thermal stresses. It is applied to describe the change in temperature.

The applied stress levels were defined as $K = 11605/tempK$, where $1/11605$ is the Boltzmann constant in electron volts per degree Celsius and $tempK = temp^oC + 273.15$ is the temperature on the Kelvin scale. The standardized stress levels (S_1, S_2) are $(0.6937, 1)$. The MLEs of the model parameters were estimated as $\hat{\lambda} = 3.74449$, $\hat{\theta}_{01} = 7.3567 \times 10^5$ and $\hat{\theta}_{02} = 3.7891 \times 10^5$ to establish any two-level CSALT in a life test. The results of the optimal transformed stress levels h_2^* under D-optimality and A-optimality criterion are provided, respectively, as 1.44154 and 1.44155.

6.2 Example 2

Another illustrative example is studied to support the proposed criterions. The data set, presented in Wu et al. [45], represents the accelerated failure times and causes with bivariate dependent competing risks model. Using the Kolmogorov-Smirnov test, the data can be fitted by the Lindely distribution with $p-$value 0.0957. In this data set, temperature is the accelerated stress. There exist three accelerated stress levels, namely $S_1 = 303K$, $S_2 = 333K$ and $S_3 = 363K$ and the normal use stress level is $S_0 = 278K$. At each stress level S_i, $n_i = 20$ units are tested for $i = 1, 2, 3$. The numbers of removals are $(r_1, r_2, r_3) = (8, 12, 16)$, $R_1 = (12, 0, 0, \ldots, 0)$, $R_2 = (8, 0, 0, \ldots, 0)$ and $R_3 = (4, 0, 0, \ldots, 0)$. Table 3 displays the data set II, that will be used in this example, under each stress level S_i for $i = 1, 2, 3$.

Table 3: Data set II.

S_1	(4.1980 1) (23.8313 1) (26.0152 2) (29.3782 1) (29.3803 1) (30.6721 1) (40.0238 2) (59.0081 2)
S_2	(5.9225 2) (6.8313 1) (8.1998 1) (11.4263 1)(11.4930 1) (12.8073 1) (15.3430 1) (17.0249 1) (18.4409 2) (18.4960 1) (22.2692 2) (35.3098 2)
S_3	(1.3674 1) (1.5807 2) (1.8340 2) (4.2250 2) (5.0150 1) (5.0246 2) (5.2322 2) (6.4447 2) (6.5251 2) (7.0332 2) (7.4556 2) (9.0729 1) (9.8901 2) (11.6041 2) (12.9922 2) (14.4516 1)

The accelerated function $\phi(S_i) = \dfrac{-1}{S_i}$ is considered to extrapolate the estimators of the unknown parameters at S_0 normal use stress level. For $h_1 = 1$, $h_2 = 2.0018$ and $h_3 = 2.83802$, MLEs of the model parameters were calculated as $\hat{\lambda} = 2.95711$, $\hat{\theta}_{01} = 0.0029456$ and $\hat{\theta}_{02} = 0.00298175$. The Fisher information matrix was obtained as,

$$I(\widehat{\lambda}, \widehat{\theta_{01}}, \widehat{\theta_{02}}) = \begin{pmatrix} 0.384899 & -0.000797421 & -0.000830053 \\ -0.000797421 & 2.0257 \times 10^{-6} & 1.71967 \times^{-6} \\ -0.000830053 & 1.71967 \times 10^{-6} & 2.15324 \times 10^{-6} \end{pmatrix}. \tag{39}$$

The optimal transformed stress levels h_2^* and h_3^* under D-optimality criterion are, respectively, equal to 1.66016 and 2.18744. For the A-optimality criterion, h_2^* and h_3^* under D-optimality criterion are, respectively, equal to 1.60079 and 1.94961.

7 Simulation Study

This simulation research is carried out using different values of n_i, m_i and R_{ij}, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, m_i$. This section investigates the performance of MLEs and BEs through mean square errors (MSEs) and averages of the estimates (AEs) in case of informative and non-informative priors. Asymptotic and credible CIs are obtained. The applied CSs in this simulation study are presented in Table 4. Table 5 shows MSEs and AEs of the MLEs and BEs when the model parameters have informative priors. MSEs and AEs of the MLEs and BEs in case of non-informative priors are introduced in Table 6. Furthermore, Table 7 includes coverage probabilities (CP) and lengths of 95% asymptotic and credible CIs in case of informative priors.

This simulation study for CSALT with competing risks for LD under PT-IIC is performed by the following algorithm:

Algorithm (3)

1. Start by determine n_i, m_i, c, k stress levels, acceleration factors $h_1, h_2, \ldots, h_k$ and the number of competing risks p.
2. For specified values of (a_1,a_2), (b_1,b_2) and (g_1,g_2) prior parameters, generate θ_{01}, θ_{02} and λ from (23), (24) and (25) respectively. For non-informative priors, specify (a_1,a_2), (b_1,b_2) and (g_1,g_2) equal zero.

3. k of random samples with size m_i are generated as $(U_{i1}, U_{i2}, \ldots, U_{im_i})$, $i = 1, 2, \cdots, k$ from Uniform$(0, 1)$ distribution.

4. Specify the censoring schemes, R_{ij}, $i = 1, 2, \ldots, k$, and $j = 1, 2, \ldots, m_i$ where $\sum_{j=1}^{m_i} R_{ij} = n_i - m_i$.

5. Let $E_{ij} = U_{ij}^{1/(j+\sum_{d=m_i-j+1}^{m_i} R_{id})}$, $j = 1, 2, \ldots, m_i$, and $i = 1, 2, \cdots, k$.

6. Procure the PT-IIC samples $(U^*_{i1}, U^*_{i2}, \ldots, U^*_{im_i})$, where $U^*_{ij} = 1 - \prod_{d=m_i-j+1}^{m_i} E_{id}$, $j = 1, 2, \ldots, m_i$ and $i = 1, 2, \ldots, k$.

7. From Step 6, generate random samples $(t_{i1}, \ldots, t_{im_i})$, $i = 1, 2, \ldots, k$, as follows:

$$\log[1 - u_{ij}] = \log[A(t_{ij})] - \log[B(t_{ij})] - \lambda^{h_i} t_{ij}(\theta_{01} + \theta_{02}), \tag{40}$$

where

$$A(t_{ij}) = 1 + (\theta_{01} + \theta_{02})\lambda^{h_i}(1 + t_{ij}) + \theta_{01}\theta_{02}\lambda^{2h_i}(1 + t_{ij})^2, \tag{41}$$

and

$$B(t_{ij}) = 1 + (\theta_{01} + \theta_{02})\lambda^{h_i} + \theta_{01}\theta_{02}\lambda^{2h_i}. \tag{42}$$

8. Determine the failure cause indicator $c_{ijp},\ i = 1, 2, \ldots, k,\ p = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, m_i$ using the ratio,

$$R = \int_0^\infty f_{ij1}(t_{ij})\ \bar{F}_{ij2}\ (t_{ij}) dt_{ij}. \tag{43}$$

9. Calculate MLEs of the three parameters depending on PT-IIC data using the solution of the nonlinear system of three equations by Newton-Raphson iteration.

10. Obtain the relative BEs to SE and LINXL functions of the parameters, using Algorithm (1), by $N = 11000$ and $M = 1000$.

11. Determine parameters of the asymptotic confidence intervals (CIs) with a confidence level of 95%.

12. Obtain the credible CIs via Algorithm (2) .

13. Repeat Steps $(3 - 11)$ for 1000 times.

14. Calculate MSEs and AEs average values resulted from MLEs and BEs.

15. Perform Steps $(1 - 13)$ using another values of n_i, m_i, R_{ij}, $j = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, k$ and the prior parameters.

16. End.

Here, we obtained Table 4.

Table 4: The PT-IIC schemes applied in the simulation studies.

CS	$(R_{i1}, \ldots, R_{im_i})$
[1]	$R_{ij} = \begin{cases} n_i - m_i, & j = 1, i = 1,2,3,4. \\ 0, & \text{otherwise}. \end{cases}$
[2]	$R_{ij} = \begin{cases} 1, & j = 1, \ldots, n_i - m_i, \quad i = 1,2,3,4. \\ 0, & \text{otherwise}. \end{cases}$
[3]	$R_{ij} = \begin{cases} n_i - m_i, & j = m_i, \quad i = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$
[4]	$R_{ij} = 0, \quad j = 0, \quad i = 1,2,3,4$

Table 5: MSE and AE of MLEs and BEs in case of informative priors under SE and LINXL function for the parameters $\lambda = 1.2$, $\theta_{01} = 0.4$ and $\theta_{02} = 0.5$.

n	m	CS		MSE				AE			
				ML	SE	LINXL		ML	SE	LINXL	
						C=−2	C=2			C=−2	C=2
$n_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	$m_i = \begin{cases} 25 & i=1, \\ 13 & i=2, \\ 11 & i=3, \\ 6 & i=4. \end{cases}$	[1]	λ	0.05297	0.00727	0.00733	0.00725	1.19345	1.13984	1.42440	1.13755
			θ_{01}	0.03159	0.00562	0.00543	0.00581	0.22944	0.33952	0.34079	0.33827
			θ_{02}	0.05574	0.01730	0.01678	0.01787	0.27247	0.41313	0.41588	0.41030
		[2]	λ	0.04942	0.00959	0.01064	0.00882	1.21419	1.16589	1.17238	1.16043
			θ_{01}	0.03386	0.00952	0.00923	0.00982	0.22282	0.32635	0.32801	0.32470
			θ_{02}	0.05924	0.02723	0.02661	0.02793	0.26498	0.37873	0.38182	0.37547
		[3]	λ	0.04206	0.01396	0.01516	0.01301	1.15899	1.15187	1.15690	1.14730
			θ_{01}	0.02928	0.01046	0.01028	0.01063	0.23791	0.32318	0.32435	0.32208
			θ_{02}	0.05264	0.02508	0.02446	0.02570	0.07876	0.38884	0.39139	0.38637
$n_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	$m_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	[4]	λ	0.05318	0.01115	0.01212	0.01036	1.22437	1.17134	1.17634	1.16679
			θ_{01}	0.03424	0.01173	0.01146	0.01200	0.22432	0.31813	0.31965	0.31664
			θ_{02}	0.06053	0.03190	0.03164	0.03217	0.26174	0.37289	0.37441	0.37136
$n_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	$m_i = \begin{cases} 42 & i=1, \\ 33 & i=2, \\ 22 & i=3, \\ 15 & i=4. \end{cases}$	[1]	λ	0.05406	0.00323	0.00353	0.00295	1.24524	1.24075	1.24406	1.23748
			θ_{01}	0.03734	0.00114	0.00126	0.00103	0.21455	0.43069	0.43242	0.42899
			θ_{02}	0.06525	0.00035	0.00033	0.00037	0.25092	0.48179	0.48235	0.48125
		[2]	λ	0.06393	0.00402	0.00436	0.00371	1.24552	1.25208	1.25513	1.24908
			θ_{01}	0.03189	0.00141	0.00154	0.00129	0.23210	0.43435	0.43610	0.43262
			θ_{02}	0.05959	0.00039	0.00037	0.00041	0.26549	0.48116	0.48172	0.48062
		[3]	λ	0.03153	0.00320	0.00351	0.00291	1.17830	1.24813	1.25112	1.14515
			θ_{01}	0.03093	0.00081	0.00089	0.00073	0.22974	0.42545	0.42691	0.42399
			θ_{02}	0.05611	0.00050	0.00048	0.00052	0.26882	0.47864	0.47917	0.47813
$n_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	$m_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	[4]	λ	0.03551	0.00415	0.00451	0.00380	1.17691	1.12534	1.12564	1.12038
			θ_{01}	0.03023	0.00133	0.00144	0.00122	0.23395	0.43281	0.43439	0.43125
			θ_{02}	0.05378	0.00042	0.00040	0.00044	0.27326	0.48190	0.48250	0.48132

Table 6: MSE and AE of MLEs and BEs in case of non-informative priors under SE and LINXL function for the parameters $\lambda = 1.2$, $\theta_{01} = 0.4$ and $\theta_{02} = 0.5$.

n	m	CS		MSE			AE		
				SE	LINXL		SE	LINXL	
					C=−2	C=2		C=−2	C=2
$n_i=\begin{cases}29 & i=1,\\ 16 & i=2,\\ 13 & i=3,\\ 7 & i=4.\end{cases}$	$m_i=\begin{cases}25 & i=1,\\ 13 & i=2,\\ 11 & i=3,\\ 6 & i=4.\end{cases}$	[1]	λ	0.00690	0.00687	0.00696	1.13794	1.14042	1.13571
			θ_{01}	0.00390	0.00370	0.00410	0.34447	0.34576	0.34322
			θ_{02}	0.02124	0.02091	0.02152	0.39078	0.39205	0.38964
		[2]	λ	0.01824	0.02003	0.01686	1.16803	1.17379	1.16290
			θ_{01}	0.01505	0.01499	0.01511	0.31062	0.31133	0.30995
			θ_{02}	0.04135	0.04104	0.04163	0.33909	0.34056	0.33763
		[3]	λ	0.05770	0.06603	0.04987	1.24397	1.25404	1.23397
			θ_{01}	0.01419	0.01403	0.01432	0.31288	0.31374	0.31210
			θ_{02}	0.03913	0.03849	0.03986	0.34386	0.34651	0.34111
$n_i=\begin{cases}29 & i=1,\\ 16 & i=2,\\ 13 & i=3,\\ 7 & i=4.\end{cases}$	$m_i=\begin{cases}29 & i=1,\\ 16 & i=2,\\ 13 & i=3,\\ 7 & i=4.\end{cases}$	[4]	λ	0.01950	0.02470	0.01490	1.18207	1.19076	1.17306
			θ_{01}	0.00612	0.00572	0.00648	0.33512	0.33667	0.33374
			θ_{02}	0.03915	0.03857	0.03961	0.34049	0.34190	0.33931
$n_i=\begin{cases}50 & i=1,\\ 40 & i=2,\\ 28 & i=3,\\ 20 & i=4.\end{cases}$	$m_i=\begin{cases}42 & i=1,\\ 33 & i=2,\\ 22 & i=3,\\ 15 & i=4.\end{cases}$	[1]	λ	0.01821	0.01948	0.01712	1.20256	1.20734	1.19809
			θ_{01}	0.02408	0.02391	0.02425	0.25929	0.26004	0.25854
			θ_{02}	0.04685	0.04685	0.04685	0.29611	0.29616	0.29607
		[2]	λ	0.01114	0.01139	0.01092	1.15572	1.15887	1.15276
			θ_{01}	0.01787	0.01780	0.01794	0.28437	0.28481	0.28392
			θ_{02}	0.04091	0.04063	0.04116	0.31682	0.31766	0.31608
		[3]	λ	0.05154	0.05368	0.04960	1.16711	1.17099	1.16342
			θ_{01}	0.01380	0.01378	0.01382	0.30576	0.30599	0.30553
			θ_{02}	0.04360	0.04359	0.04360	0.32097	0.32110	0.32085
$n_i=\begin{cases}50 & i=1,\\ 40 & i=2,\\ 28 & i=3,\\ 20 & i=4.\end{cases}$	$m_i=\begin{cases}50 & i=1,\\ 40 & i=2,\\ 28 & i=3,\\ 20 & i=4.\end{cases}$	[4]	λ	0.02078	0.02206	0.01961	1.20598	1.20986	1.20227
			θ_{01}	0.02871	0.02870	0.02873	0.24396	0.24406	0.24386
			θ_{02}	0.04632	0.04570	0.04692	0.29708	0.29875	0.29556

Table 7: Lengths and CP of 95% asymptotic and credible CIs for Informative Priors (IF) and Non-Informative Priors (NIF) for the parameters $\lambda = 1.2$, $\theta_{01} = 0.4$ and $\theta_{02} = 0.5$.

n	m	CS		Lengths: Asymptotic CI	Lengths: Credible CI(IF)	Lengths: Credible CI(NIF)	CP: Asymptotic CI	CP: Credible CI(IF)	CP: Credible CI(NIF)
$n_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	$m_i = \begin{cases} 25 & i=1, \\ 13 & i=2, \\ 11 & i=3, \\ 6 & i=4. \end{cases}$	[1]	λ	2.46335	0.15030	0.13467	1	0.62	0.6
			θ_{01}	0.62327	0.10938	0.10963	0.88	0.72	0.6
			θ_{02}	0.74597	0.13798	0.08479	0.72	0.78	0.5
		[2]	λ	2.51965	0.21684	0.21032	1	0.72	0.8
			θ_{01}	0.62064	0.12109	0.08420	0.78	0.7	0.5
			θ_{02}	0.73691	0.11863	0.10477	0.7	0.6	0.4
		[3]	λ	2.30097	0.17763	0.26727	1	0.64	0.4
			θ_{01}	0.63474	0.19381	0.09971	0.82	0.72	0.6
			θ_{02}	0.73838	0.11123	0.09228	0.76	0.66	0.5
$n_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	$m_i = \begin{cases} 29 & i=1, \\ 16 & i=2, \\ 13 & i=3, \\ 7 & i=4. \end{cases}$	[4]	λ	2.69409	0.19715	0.22267	1	0.62	0.7
			θ_{01}	0.63662	0.10983	0.11194	0.8	0.65	0.6
			θ_{02}	0.74478	0.08928	0.06764	0.7	0.6	0.4
$n_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	$m_i = \begin{cases} 42 & i=1, \\ 33 & i=2, \\ 22 & i=3, \\ 15 & i=4. \end{cases}$	[1]	λ	1.89007	0.21079	0.1985	1	0.9	0.4
			θ_{01}	0.46634	0.16457	0.04891	0.5	1	1
			θ_{02}	0.54089	0.08029	0.00905	0.6	1	1
		[2]	λ	1.77117	0.19823	0.17934	1	0.8	0.6
			θ_{01}	0.45115	0.16118	0.04954	0.62	1	0.3
			θ_{02}	0.51464	0.08541	0.03395	0.6	1	0.3
		[3]	λ	1.64296	0.20786	0.14751	1	0.96	0.2
			θ_{01}	0.44357	0.14704	0.04556	0.7	1	0.1
			θ_{02}	0.51413	0.08093	0.02086	0.6	1	0.2
$n_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	$m_i = \begin{cases} 50 & i=1, \\ 40 & i=2, \\ 28 & i=3, \\ 20 & i=4. \end{cases}$	[4]	λ	1.69542	0.20821	0.18778	1	0.88	0.6
			θ_{01}	0.46750	0.15372	0.01936	0.68	1	0.1
			θ_{02}	0.54626	0.08541	0.03315	0.66	0.96	0.1

8 Conclusion

This article discusses the inference of the CSALT in the presence of LD competing risks under PT-IIC. Via simulation studies, MLEs and BEs are produced in the both cases of prior distributions (informative and non-informative) for the model parameters θ_{01}, θ_{02} and λ. The informative priors have favorable statistical features. Asymptotic and credible CIs are conducted for the parameters. Calculations have been performed by several progressive CSs with different sample sizes. It is obvious that the statistical computations will be more expensive when we study more failure causes. Based on the results in Tables 5–7, we have concluded that BEs is a preferable choice to get better outcomes through the simulation studies. In addition, lengths of approximate and credible CIs decrease when the sample size increase. With a high sample size, credible CIs deliver more precise conclusions through CP than approximate CIs.

A comparison between the estimations using informative and non-informative priors is performed to demonstrate that the informative prior produces superior outcomes. The numerical results supports clearly the theoretical assumptions. The D-optimality and A-optimality Criterion are applied successfully with satisfying results.

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